

Exact Momentum Distribution of a Fermi Gas in One Dimension

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We introduce an exactly solvable model of a fermi gas in one dimension and compute the momentum distribution exactly. This is based on a generalisation of the ideas of bosonization in one dimension. It is shown that in the RPA limit (the ultra-high density limit) the answers we get are the exact answers for a homogeneous fermi gas interacting via a two-body repulsive coulomb interaction. Furthermore, the solution may be obtained exactly for arbitrary functional forms of the interaction, so long as it is purely repulsive. No linearization of the bare fermion dispersion is required. We find that for the interaction considered, the fermi surface is intact for weak repulsion and is destroyed only for sufficiently strong repulsion. Comparison with other models like the supersymmetric t-J model with inverse square interactions is made.

Recent years have seen remarkable developments in many-body theory in the form an assortment of techniques that may be loosely termed bosonization. The beginnings of these types of techniques may be traced back to the work of Tomonaga [1] and later on by Luttinger [2] and by Lieb and Mattis [3]. The work of Sawada [4] and Arponen and Pajanne [5] in recasting the fermi gas problem in a bose language has to be mentioned. Arponen and Pajanne recover corrections to the Random Phase Approximation (RPA) of Bohm and Pines [6] in a systematic manner. In the 70's an attempt was made by Luther [8] at generalising these ideas to higher dimensions. Closely related to this is work by Sharp et. al. [15] in current algebra. More progress was made by Haldane [9] which culminated in the explicit computation of the single particle propagator by Castro-Neto and Fradkin [10] and by Houghton, Marston et.al. [11] and also by Kopietz et. al. [12]. Rigorous work by Frohlich and Marchetti [13] is also along similar lines. Also the work of Frau et. al. [14] on algebraic bosonization is relevant to the present Letter as the authors have considered effects beyond the linear dispersion in that article.

In this Letter, we try to use the formalism introduced by us [18] in an earlier preprint to write down an interaction [19], [20] that closely mimics the two-body repulsive interaction between the fermions and compute the momentum distribution exactly. It is also shown that in the ultra-high density limit, the interaction is exactly the two-body repulsive coulomb interaction insofar as it is able to recover the random phase approximation exactly. This momentum distribution differs from the traditional Luttinger model in one dimension in this aspect, namely, the discontinuity at the fermi momentum is present for weak repulsion and is destroyed only for stronger repulsions. But the present model does share similarities with other models

such as the supersymmetric t-J model [17] which does show a discontinuity at the fermi momentum in its momentum distribution.

In the sissa preprint [18] we wrote down a correspondence between products of fermi fields and bose fields that represent the displacements of the fermi sea, in a manner analogous to the fermi-surface displacements introduced by Haldane [9]. There, the bose fields in question were written as $a_{\mathbf{k}}(\mathbf{q})$ and $a_{\mathbf{k}}^\dagger(\mathbf{q})$. Linear combination of these operators would then correspond to displacement operators of the fermi sea at wavevector \mathbf{k} by an amount \mathbf{q} . This is a generalisation of the concept of fermi-surface displacements where the magnitude of \mathbf{k} would be restricted to equal k_f the fermi momentum. This enables us to draw up a correspondence as described in the sissa preprint [18]. The boldface on the vectors \mathbf{q} and \mathbf{k} serve to illustrate that these ideas admit a straightforward generalisation to more than one dimension. Let $c_{\mathbf{k}}$ and $c_{\mathbf{k}}^\dagger$ be the fermi fields in question, then,

$$\begin{aligned} c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{k}-\mathbf{q}/2} &= n_F(\mathbf{k}) \frac{N}{\langle N \rangle} \delta_{\mathbf{q},0} + \left(\sqrt{\frac{N}{\langle N \rangle}} \right) [\Lambda_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(-\mathbf{q}) + \Lambda_{\mathbf{k}}(-\mathbf{q}) a_{\mathbf{k}}^\dagger(\mathbf{q})] \\ &+ \sum_{\mathbf{q}_1} \Lambda_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}(-\mathbf{q}_1) \Lambda_{\mathbf{k}-\mathbf{q}_1/2}(\mathbf{q}-\mathbf{q}_1) a_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}^\dagger(\mathbf{q}_1) a_{\mathbf{k}-\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1) \\ &- \sum_{\mathbf{q}_1} \Lambda_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2}(-\mathbf{q}_1) \Lambda_{\mathbf{k}+\mathbf{q}_1/2}(\mathbf{q}-\mathbf{q}_1) a_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2}^\dagger(\mathbf{q}_1) a_{\mathbf{k}+\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1) \end{aligned} \quad (1)$$

Here, $a_{\mathbf{k}}(\mathbf{q})$ and $a_{\mathbf{k}}^\dagger(\mathbf{q})$ are exact bose operators and $c_{\mathbf{k}}$ and $c_{\mathbf{k}}^\dagger$ are exact fermion operators. In symbols,

$$\{c_{\mathbf{k}}, c_{\mathbf{k}'}\} = 0; \quad \{c_{\mathbf{k}}, c_{\mathbf{k}'}^\dagger\} = \delta_{\mathbf{k},\mathbf{k}'} \quad (2)$$

and,

$$[a_{\mathbf{k}}(\mathbf{q}), a_{\mathbf{k}'}(\mathbf{q}')] = 0; \quad [a_{\mathbf{k}}(\mathbf{q}), a_{\mathbf{k}'}^\dagger(\mathbf{q}')] = \delta_{\mathbf{k},\mathbf{k}'} \delta_{\mathbf{q},\mathbf{q}'} \quad (3)$$

and, $\Lambda_{\mathbf{k}}(\mathbf{q}) = \sqrt{n_F(\mathbf{k} + \mathbf{q}/2)(1 - n_F(\mathbf{k} - \mathbf{q}/2))}$ and, $n_F(\mathbf{k}) = \theta(k_f - k)$. Furthermore, the kinetic energy operator is given in the fermi language as,

$$K = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} c_{\mathbf{k}}^\dagger c_{\mathbf{k}} \quad (4)$$

where $\epsilon_{\mathbf{k}} = k^2/2m$. The same operator has an extremely elegant form in the bose language,

$$K = E_0 + \sum_{\mathbf{k},\mathbf{q}} \omega_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}^\dagger(\mathbf{q}) a_{\mathbf{k}}(\mathbf{q}) \quad (5)$$

here, $E_0 = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} n_F(\mathbf{k})$, and the excitation energy is given by,

$$\omega_{\mathbf{k}}(\mathbf{q}) = \left(\frac{\mathbf{k} \cdot \mathbf{q}}{m} \right) \Lambda_{\mathbf{k}}(-\mathbf{q}) \quad (6)$$

Also the number operator $N = \sum_{\mathbf{k}} c_{\mathbf{k}}^\dagger c_{\mathbf{k}}$ is to be distinguished from its expectation value, $\langle N \rangle = \sum_{\mathbf{k}} n_F(\mathbf{k})$. The coefficient $\Lambda_{\mathbf{k}}(-\mathbf{q})$ is non-zero only when $\mathbf{k} \cdot \mathbf{q} \geq 0$. The meaning of the above formula in Eq.(1) is as follows. All the dynamical moments of $c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{k}+\mathbf{q}/2}$ evaluated in the fermi language are going to be identical to those evaluated using the bose representation of this product, provided, we identify the noninteracting fermi-sea with the bose vacuum. In other words,

$$a_{\mathbf{k}}(\mathbf{q})|FS\rangle = 0 \quad (7)$$

From this it is possible to recast any problem involving interacting fermions in a form involving only bose fields. In particular, if we selectively choose parts of the interaction terms written out in the bose language, it may be possible to compute exactly, important quantities such as, momentum distribution, dynamical density correlation functions and so on. Let us now focus on the homogeneous fermi gas. The interaction has the form,

$$U = \sum_{\mathbf{q} \neq 0} \frac{v_{\mathbf{q}}}{2V} (\rho_{\mathbf{q}} \rho_{-\mathbf{q}} - N) \quad (8)$$

Written out in terms of the bose fields, it has a part that is quadratic in the bose fields, and we have cubic and quartic terms as well. Let us just focus on the part that is quadratic in the bose fields. This amounts to postulating a phenomenological interaction that is not exactly of the two-body type (in the original fermi system) but something that mimics it. How closely does it mimic this two-body interaction, is the big question. This question may be answered by computing the dielectric function and demonstrating that it is exactly equal to the RPA dielectric function. This has been done in our important preprint that computes the single-particle green functions of the same system [20]. Thus our results are the exact results for a homogeneous fermi gas with two-body repulsive interactions in the same limit in which the RPA is exact. Written out in the bose language this interaction has the form,

$$H_I = \sum_{\mathbf{q} \neq 0} \frac{v_{\mathbf{q}}}{2V} \sum_{\mathbf{k}, \mathbf{k}'} [\Lambda_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(-\mathbf{q}) + \Lambda_{\mathbf{k}}(-\mathbf{q}) a_{\mathbf{k}}^\dagger(\mathbf{q})] [\Lambda_{\mathbf{k}'}(-\mathbf{q}) a_{\mathbf{k}'}(\mathbf{q}) + \Lambda_{\mathbf{k}'}(\mathbf{q}) a_{\mathbf{k}'}^\dagger(-\mathbf{q})] \quad (9)$$

The full hamiltonian is now $H = K + H_I$. This hamiltonian may be diagonalised exactly and the momentum distribution may be computed. Alternatively, the equation of motion method may be used to obtain the same answers. In fact, we have used both these methods [20] to obtain the exact answer for the momentum distribution in one dimension (the formulas written down till now are valid in any number of dimensions).

The exact momentum distribution found by us has the form ($\hbar = c = 1$),

$$\begin{aligned} \langle c_k^\dagger c_k \rangle &= n_F(k) + (2\pi k_f) \int_{-\infty}^{+\infty} \frac{dq_1}{2\pi} \frac{\Lambda_{k-q_1/2}(-q_1)}{2\omega_R(q_1)(\omega_R(q_1) + \omega_{k-q_1/2}(q_1))^2 (\frac{m^3}{q_1^4})(\cosh(\lambda(q_1)) - 1)} \\ &- (2\pi k_f) \int_{-\infty}^{+\infty} \frac{dq_1}{2\pi} \frac{\Lambda_{k+q_1/2}(-q_1)}{2\omega_R(q_1)(\omega_R(q_1) + \omega_{k+q_1/2}(q_1))^2 (\frac{m^3}{q_1^4})(\cosh(\lambda(q_1)) - 1)} \end{aligned} \quad (10)$$

$$\lambda(q) = (\frac{2\pi q}{m})(\frac{1}{v_q}) \quad (11)$$

$$\omega_R(q) = (\frac{|q|}{m}) \sqrt{\frac{(k_f + q/2)^2 - (k_f - q/2)^2 \exp(-\lambda(q))}{1 - \exp(-\lambda(q))}} \quad (12)$$

The above answer for the momentum distribution Eq.(10) is the exact momentum distribution of a homogeneous fermi gas in one dimension interacting via a two-body repulsive interaction. As has been pointed out earlier, the RPA dielectric function as described by Bohm and Pines [6] may be recovered exactly by this model provided one identifies the density operator with the object obtained by retaining only terms linear in the sea-displacements.

$$\rho_{\mathbf{q}} = \sum_{\mathbf{k}} [\Lambda_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(-\mathbf{q}) + \Lambda_{\mathbf{k}}(-\mathbf{q}) a_{\mathbf{k}}^{\dagger}(\mathbf{q})] \quad (13)$$

The above form ignores the terms quadratic in the sea-displacements. The justification being that this definition along with the standard procedure for obtaining the dielectric function, namely, add a periodic weak potential coupled with the density operator above and compute the ratio of the external versus the effective potential, gives the exact RPA dielectric function thereby suggesting that dropping terms quadratic in the displacements (in the definition of the density) is not going to be important in the same limit in which the RPA itself is exact (for a pedagogical description of this procedure of obtaining the dielectric function please consult the text by Kadanoff and Baym [7]). In order to see how good the present theory is, it is desirable to compare these results with the Calogero-Sutherland model or more specifically with the spin-spin correlation function of the Haldane-Shastry model. This is given by Lesage et. al. [16], Here they find that the momentum distribution is smooth and does not possess any discontinuity or kinks. This is in contrast with the solution above (Eq.(10)), where we see the presence of a fermi surface for sufficiently weak repulsion. However, the supersymmetric t-J model [17] with inverse square interactions does possess a jump at the fermi surface, and therefore the present model belongs to this latter class. Also, the system does not exhibit Wigner crystallization at low density. This is another drawback of the interaction that we have considered. But this is to be expected since the present model captures the two-body repulsive interaction only in the ultra-high density limit. On the plus side, the highly nonperturbative nature of the solution suggested by the presence of the term, $\cosh(\frac{2\pi q}{m} \frac{1}{v_q}) - 1$ cannot be missed.

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